

Matching of nonthermal initial conditions and hydrodynamic stage in ultrarelativistic heavy-ion collisions

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Abstract

A simple approach is proposed allowing actual calculations of the preequilibrium dynamics in ultrarelativistic heavy-ion collisions to be performed for a far-from-equilibrium initial state. The method is based on the phenomenological macroscopic equations that describe the relaxation dynamics of the energy-momentum tensor and are motivated by Boltzmann kinetics in the relaxation-time approximation. It gives the possibility to match smoothly a nonthermal initial state to the hydrodynamics of the quark gluon plasma. The model contains two parameters, the duration of the prehydrodynamic stage and the initial value of the relaxation-time parameter, and allows one to assess the energy-momentum tensor at a supposed time of initialization of the hydrodynamics.

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I. INTRODUCTION

A comprehensive analysis of the experimental data from the Relativistic Heavy-Ion Collider (RHIC) has shown that a quark-gluon plasma (QGP) is created in these collisions, and that the thermalized QGP is, perhaps, the most perfect liquid possible in nature. This conclusion is based, in particular, on the success of ideal hydrodynamics in describing of the basic features of heavy-ion collisions at RHIC energies (for a review see, e.g., Ref. [1]). Recently, essential progress has been made in the development and applications of viscous hydrodynamics for RHIC heavy-ion collisions (for reviews and recent results see, e.g., [1–3] and references therein). Viscous hydrodynamics accounts for deviations from local equilibrium by means of dissipative transport coefficients, and, therefore, has a more extended region of applicability as compared to ideal hydrodynamics. Nevertheless, it should be emphasized that the domain of validity of viscous hydrodynamics is still the hydrodynamic regime; thus hydrodynamics is valid when the relaxation time τ_{rel} is much smaller than the inverse expansion rate, $1/\partial_\mu u^\mu$, that is, $\tau_{\text{rel}}\partial_\mu u^\mu \ll 1$, and one can expect that hydrodynamics breaks down when $\tau_{\text{rel}}\partial_\mu u^\mu \sim 1/2$. In practice, hydrodynamics breaks down in ultrarelativistic heavy-ion collisions at the very initial nonequilibrium stage, near the edge of the fireball, and at the later rarefied kinetic stage of matter evolution. Since hydrodynamics is based on the assumption that the system is near local thermal equilibrium [4–6], it is natural that its region of applicability cannot be reliably determined from within itself, and can be properly estimated only with the help of an appropriate nonequilibrium theory.

In addition to the breakdown times, one also needs to specify initial conditions, such as the energy density, fluid velocities, and viscous shear tensor to apply hydrodynamics. Evidently, the initial conditions for hydrodynamics are determined at the very initial nonequilibrium stage of the matter evolution in ultrarelativistic heavy-ion collisions. This stage is well understood now based on the color glass condensate (CGC) approach (for a review see, e.g., Ref. [7]), which is adequate at RHIC and, probably, Large Hadron Collider (LHC) energies, although an explanation of the thermalization and nearly perfect fluidity provides a challenge for the theory of quark-gluon matter (for recent results, see Refs. [8, 9]). It is worth noting, however, that the assumption of very early thermalization (say, $\tau_{\text{th}} = 0.3 \text{ fm}/c$) is not necessary for data description in hydrodynamics, since the transverse collective flows and their azimuthal anisotropy for noncentral collisions can appear already at the beginning

of the hydrodynamic expansion as a result of the development of the transverse velocities at the prethermal (glasma [10] or partonic or string [11]) stage [12]. That is why an analysis of matter evolution at the prethermal stage, which determines the further hydrodynamic expansion, is so important..

The nonthermal initial conditions are related typically to the Bjorken proper time τ_0 when the system can be characterized by the phase-space density of individual partons. In the CGC approach this "formation time" was estimated to be $\tau_0 \approx 0.1 - 0.3$ fm/c for central collisions in midrapidity at LHC and RHIC energies [13, 14]. Taking into account the theoretical estimate of the thermalization time scale $\tau_{\text{th}} = 1 - 1.5$ fm/c for the LHC and RHIC heavy-ion collisions [9], one can hardly expect hydrodynamics to be applicable at $\tau = \tau_0 < 1$ fm/c, because then $\tau_{\text{rel}} \partial_\mu u^\mu \sim \tau_{\text{th}}/\tau > 1$.

Thus, to obtain the initial conditions for hydrodynamics at $\tau = \tau_{\text{th}} \gtrsim 1$ fm/c, one needs to match the very early initial stage of the nuclei collisions, when the hydrodynamic approximation is outside the regime of its validity, with an almost local equilibrium state of the QGP. It is noteworthy that phase-space distributions derived from the CGC approach are highly anisotropic, and so the distribution functions are quite nonequilibrium. The longitudinal momentum distribution is much more narrow than transverse one, and asymptotically, at late times, it is a delta function $\delta(p_z)$ at $z = 0$ [15]. The latter approximation is often used for a description of the initial stage in $A + A$ collisions (see, e.g., [16]).¹

In this paper, we formulate phenomenological macroscopic equations accounting for the energy-momentum and quantum number conservation laws which allow one to connect the arbitrary initial, probably highly nonequilibrium state of the system with the (partially) equilibrated thermal state, when one can start to use (viscous) hydrodynamics to describe the further matter evolution.² The model exploits a minimal set of phenomenological parameters and can be utilized to assess the initial conditions for hydrodynamic expansion, which can

¹ One should understand, however, that such a utilization of the asymptotic approximation is not quite correct from the point of view of quantum theory. Indeed, the Wigner function $f_W(x, p)$, which is the quantum mechanical analog of the classical phase-space density $f(x, p)$, satisfies the restriction $\int f_W^2(x, p) d^3p d^3x \leq (2\pi\hbar)^{-3}$ (see, e.g., [17]): here the normalization condition $\int f_W(x, p) d^3p d^3x = 1$ is supposed. Then, in order to escape a contradiction with quantum mechanics, smooth boost-invariant prescriptions for the longitudinal part of the distribution $f(x, p)$ from Refs. [18] or [19] should be used for a description of the initial stage in $A + A$ collisions.

² A similar effort was undertaken in Ref. [20], where the preequilibrium dynamics of a $(0 + 1)$ -dimensional QGP was matched to second-order viscous hydrodynamics.

be used, then, for a hydrodynamic modeling of ultrarelativistic heavy-ion collisions.

II. ENERGY AND MOMENTUM RELAXATION DYNAMICS FOR THE EARLY STAGE OF ULTRARELATIVISTIC HEAVY-ION COLLISIONS

It is well known that statistical derivation of the macroscopic hydrodynamic equations for gases is based on an approximate solution of the Boltzmann equations near the local equilibrium distribution (see, e.g., Refs. [4–6]). The form of hydrodynamic equations derived can then be spread out phenomenologically from a gas to dense fluids if they are near local thermal equilibrium. More precisely, hydrodynamic equations for dense systems like liquids near local thermal equilibrium can be derived by means of the (quasi) equilibrium statistical operator method based on the conditions of a maximum of local entropy and under appropriate auxiliary conditions [6]. The proximity to local thermal equilibrium allows one to utilize for a description of these inhomogeneous evolutionary systems the Gibbs thermodynamic relations in the same form as for equilibrium systems [5], and to get the equations of viscous hydrodynamics in the closed form. However, if the system is far from local equilibrium, it can be described macroscopically only through the unclosed hierarchical structure of the hydrodynamic balance equations expressing the conservation laws in the system [5]. All the momenta of the physical values are present there, and they cannot be calculated without knowledge of the evolution of the nonequilibrium distribution function [4]. Although the second-order viscous hydrodynamic equations increase precision compared with the first-order equations and, moreover, have attractive features from the point of view of mathematical formulation, the domain of applicability of these equations is still the hydrodynamic domain: the microscopic scales like the mean free path are much smaller than the macroscopic dimensions (the system's homogeneity lengths). Because of this, one can conjecture that dissipative fluid dynamics is inapplicable in its standard form to the very initial stage, $\tau < 1$ fm, of ultrarelativistic heavy-ion collisions (see also Ref. [21]), when the rate of expansion is too high and the interactions are not strong enough to validate the application of the hydrodynamic approach. Thereby, the very initial nonthermal state (glasma?) of matter evolution in $A + A$ collisions cannot be described hydrodynamically,

unlike the subsequent thermal QGP stage, where hydrodynamics seems to work well [1].³

To perform the heuristic arguments to derive phenomenological macroscopic equations for a description of transient prehydrodynamic behavior, let us first assume that the initial phase-space density is associated with the partons, while the resulting expressions can be used as a phenomenological approach not only for partonic matter but also for other forms of prethermal matter like glasma and strings. Usually, if the exact dynamics is cumbersome or unknown, establishment of the local equilibrium state in the system is modeled by the kinetic equation for the phase-space distribution function $f(x, p)$ in the relaxation time approximation, with $f^{\text{leq}}(x, p)$ as the target function to which the phase-space distribution tends. The corresponding equation has the form

$$\frac{p^\mu \partial f(x, p)}{\partial x^\mu} = -p^\mu u_\mu \frac{f(x, p) - f^{\text{leq}}(x, p)}{\tau_{\text{rel}}^*(x, p)}, \quad (1)$$

where τ_{rel}^* is the relaxation-time parameter in the local rest frame of the energy flow (in general it is some function of (x, p)), and $u^\mu(x)$ is the four-vector energy flow field,

$$u^\mu = \frac{T^{\mu\nu} u_\nu}{T^{\mu\nu} u_\mu u_\nu} = \frac{T^{\mu\nu} u_\nu}{\epsilon}, \quad (2)$$

$$T^{\mu\nu}(x) = \int d^3p \frac{p^\mu p^\nu}{p_0} f(x, p). \quad (3)$$

To determine the parameters (temperature, etc.) that define the local equilibrium state, $f^{\text{leq}}(x, p)$, it is necessary to implement the energy-momentum conservation equations,

$$\partial_\mu T^{\mu\nu}(x) = 0, \quad (4)$$

and, if necessary, the conservation equations for (net) quantum numbers q_i ,

$$\partial_\mu q_i^\mu(x) = 0, \quad (5)$$

with current $q_i^\mu(x)$,

$$q_i^\mu(x) = \int \frac{d^3k}{k_0} k^\mu [f_{q_i}(x, k) - f_{-q_i}(x, k)]. \quad (6)$$

³ Note that the results of viscous hydrodynamic calculations depend on exact values of the transport coefficients that still cannot be derived unambiguously from the complicated theory of microscopic dynamics. At the later, low-density gaslike stage, the viscous hydrodynamics can be coupled to the Boltzmann kinetics of the hadron resonance gas [22].

Then Eqs. (4) and (5) together with the formal solution of Eq. (1),

$$f(t, \mathbf{r}, p) = f(t_0, \mathbf{r} - \frac{\mathbf{p}}{p_0}(t - t_0), p)P(t_0, t, \mathbf{r}, p) + \int_{t_0}^t f^{\text{leq}}(t', \mathbf{r} - \frac{\mathbf{p}}{p_0}(t - t'), p) \frac{d}{dt'} P(t', t, \mathbf{r}, p) dt', \quad (7)$$

where the probability for particle with momentum \mathbf{p} to propagate freely from point (t', \mathbf{r}) to point $(t, \mathbf{r} + \frac{\mathbf{p}}{p_0}(t - t'))$ is

$$P(t', t, \mathbf{r}, p) = \exp \left\{ - \int_{t'}^t \frac{p_\mu u^\mu(s, \mathbf{r} - \frac{\mathbf{p}}{p_0}(t - s))}{p_0 \tau_{\text{rel}}^*(s, \mathbf{r} - \frac{\mathbf{p}}{p_0}(t - s), p)} ds \right\}, \quad (8)$$

lead to very complicated equations for the thermal and hydrodynamic fields (temperature, collective velocities, etc.) that define the target local equilibrium state described by $f^{\text{leq}}(x, p)$. The problem becomes even more severe if one takes into account that τ_{rel}^* is a function of f^{leq} . Also note that the target state is reached in a finite time interval at $t = t_{\text{th}}$,⁴ only if the relaxation-time parameter in Eq. (1) vanishes at $t \rightarrow t_{\text{th}}$: $\tau_{\text{rel}}^*(t \rightarrow t_{\text{th}}, \mathbf{r}, p) \rightarrow 0$.

All this makes difficult a utilization of Eq. (1) for a matching of a far-from-equilibrium initial state to perfect or viscous hydrodynamics with presupposed transport coefficients⁵ in relativistic heavy-ion collisions. The situation reflects the fact that the solutions of the Boltzmann equations as a rule do not correspond to any hydrodynamics, or, in some conditions (in the vicinity of a local equilibrium state), correspond to solutions of viscous hydrodynamic equations with a very specific form of the viscosity coefficients defined by the cross sections or by the relaxation-time parameters in the relaxation-time approximation of the collision terms. To utilize this in practice a quite complicated Chapman-Enskog method (see, e.g., [4, 23]) is applied. This method is inapplicable, however, if the system is initially nonthermal.

Here we propose a simple phenomenological method allowing one to describe the relaxation of an initially far-from-equilibrium system toward the neighborhood of a local equilibrium state. The latter is described by the target energy-momentum tensor and is associated,

⁴ Of course, the same statement is easy to reformulate for the case when one operates with an arbitrary three-dimensional spacelike "thermalization" hypersurface where $t = t_{\text{th}}(\mathbf{r})$ instead of a constant time t_{th} .

⁵ Then the target function in Eq. (1) could have a nonequilibrium form with parameters that are linked to transport coefficients.

in general, with viscous hydrodynamics. For simplicity, we will consider in detail a model with the target energy-momentum tensor associated with a perfect fluid; a generalization for the viscous target energy-momentum tensor is straightforward. To arrive at such a model, it is useful to start again with the relaxation-time approximation (1) which has the formal solution (7), (8). Then, assuming that the maximum of the integrand in (7) occurs at the upper limit of the integral and, moreover, that $f^{\text{leq}}(x, p)$ is a relatively smooth function, one can factor out from the integral in (7) the function $f^{\text{leq}}(t', \mathbf{r} - \frac{\mathbf{p}}{p_0}(t - t'), p)$ at point $t' = t$ and arrive at the simple form,

$$f(x, p) = f(t_0, \mathbf{r} - \frac{\mathbf{p}}{p_0}(t - t_0), p)P(t_0, t, \mathbf{r}, p) + f^{\text{leq}}(t, \mathbf{r}, p)(1 - P(t_0, t, \mathbf{r}, p)), \quad (9)$$

which demonstrates an explicit transition to a local equilibrium state at $t = t_{\text{th}}$ if $P(t_0, t_{\text{th}}, \mathbf{r}, p) = 0$ and is, of course, just an approximation of (7). Let us write this expression in some more general relativistic invariant form, aiming to use the curvature coordinates in our further analysis,

$$f(x, p) = f_{\text{free}}(x, p)\mathcal{P}(x, p) + f^{\text{leq}}(x, p)(1 - \mathcal{P}(x, p)), \quad (10)$$

where $f_{\text{free}}(x, p)$ is the distribution function of freely streaming partons,

$$p^\mu \partial_\mu f_{\text{free}}(x, p) = 0, \quad (11)$$

which coincides with $f(x, p)$ at some three-dimensional spacelike hypersurface where initial conditions for $f(x, p)$ are formulated, for example, $f_{\text{free}}(x, p) = f(t_0, \mathbf{r} - \frac{\mathbf{p}}{p_0}(t - t_0), p)$ for the initial distribution function $f(t_0, \mathbf{r}, p)$. Here $\mathcal{P}(x, p)$ has the same probability interpretation as P : it is the probability for the particle emitted on the initial hypersurface (in simplest case - at time t_0) with momentum \mathbf{p} to reach point x :

$$\mathcal{P}(x, p) = \exp \left\{ - \int d^4x' G_p(x - x') \frac{p^\mu u_\mu(x')}{\tau_{\text{rel}}^*(x', p)} \right\}, \quad (12)$$

$$p^\mu \partial_\mu G_p(x - x') = \delta^{(4)}(x - x'), \quad (13)$$

$\mathcal{P} = 1$ initially at $t = t_0$ (or at some three-dimensional hypersurface $t_0(\mathbf{r})$ where initial conditions are specified), $\mathcal{P} = 0$ at $t = t_{\text{th}}$ (or at some hypersurface $t_{\text{th}}(\mathbf{r})$) when $\tau_{\text{rel}}^* \rightarrow 0$. One can easily see that (10) satisfies the equation

$$\frac{p^\mu \partial f(x, p)}{\partial x^\mu} = -p^\mu u_\mu \frac{f(x, p) - f^{\text{leq}}(x, p)}{\tau_{\text{rel}}^*(x, p)} + \frac{p^\mu \partial f^{\text{leq}}(x, p)}{\partial x^\mu} (1 - \mathcal{P}(x, p)). \quad (14)$$

Note here that, while Eq. (10) is relatively simple, the equations for $f^{\text{leq}}(x, p)$ that follow from conservation laws (4) and (5) are, in general, very complicated (even if one does not relate τ_{rel}^* with $f^{\text{leq}}(x, p)$) because of the momentum dependence of $\mathcal{P}(x, p)$.

Our idea is to utilize the approximate formal solution (10), which preserves the most important properties of the true dynamics at the prethermal stage of evolution, to describe the relaxation dynamics of the energy-momentum tensor. To do it, we first make the approximations

$$\tau_{\text{rel}}^*(x, p) \approx \tau_{\text{rel}}^*(x), \quad \mathcal{P}(x, p) \approx \mathcal{P}(x), \quad (15)$$

which significantly simplifies modeling of the relaxation dynamics at the early prethermal stage of evolution in ultrarelativistic heavy-ion collisions,⁶ because then, in particular, the energy-momentum tensor (3) takes the simple form

$$T^{\mu\nu}(x) = T_{\text{free}}^{\mu\nu}(x)\mathcal{P}(x) + T_{\text{hyd}}^{\mu\nu}(x)(1 - \mathcal{P}(x)), \quad (16)$$

$$T_{\text{free}}^{\mu\nu}(x) = \int d^3p \frac{p^\mu p^\nu}{p_0} f_{\text{free}}(x, p), \quad (17)$$

$$T_{\text{hyd}}^{\mu\nu}(x) = \int d^3p \frac{p^\mu p^\nu}{p_0} f^{\text{leq}}(x, p). \quad (18)$$

Even though (16) is a rather rough approximation for the true energy-momentum tensor, it preserves the desired properties of the true expression, namely, it demonstrates a transition from a far-from-equilibrium state to a local equilibrium state and, which is very important, it allows one to account for the energymomentum conservation laws and, if necessary, quantum number conservation laws in a simple form. That is, accounting for the energy-momentum conservation laws $\partial_\mu T^{\mu\nu}(x) = 0$, with $T^{\mu\nu}(x)$ defined in (16), and taking into account that $\partial_\mu T_{\text{free}}^{\mu\nu}(x) = 0$, we arrive to the following equations:⁷

$$\partial_\mu [(1 - \mathcal{P}(x))T_{\text{hyd}}^{\mu\nu}(x)] = -T_{\text{free}}^{\mu\nu}(x)\partial_\mu \mathcal{P}(x). \quad (19)$$

⁶ The example of explicit representation of $\mathcal{P}(x)$ will be given in the next section in hyperbolic Bjorken coordinates for constant longitudinal proper times corresponding to initial and final ("thermalization") hypersurfaces.

⁷ Note here, to avoid misunderstanding, that while the expression (16) looks like a smooth interpolation between the free-streaming and hydrodynamic regimes, it does not mean that initially there are no interactions and then interactions gradually switch on. One can see that this is not the case from the fact that the expression (16) is based on the approximate formal solution (10) of kinetic equation (1).

The initial conditions for $T_{\text{hyd}}^{\mu\nu}$ follow from the right-hand side of Eq. (14). That is, taking into account approximation (15) and utilizing the fact that initially $\mathcal{P} = 1$, we see that the initial conditions are given by the equation

$$T^{\mu\nu}(x_{\text{in}})u_{\mu}(x_{\text{in}}) = T_{\text{hyd}}^{\mu\nu}(x_{\text{in}})u_{\mu}(x_{\text{in}}), \quad (20)$$

where $x_{\text{in}} \equiv (t_0(\mathbf{r}), \mathbf{r})$. Note that the same equations defining the initial conditions for $T_{\text{hyd}}^{\mu\nu}$ follow from Eq. (1). The solution of Eq. (20) is straightforward and follows from the definition (2):

$$\epsilon_{\text{hyd}}(x_{\text{in}}) = \epsilon(x_{\text{in}}), \quad u_{\text{hyd}}^{\mu}(x_{\text{in}}) = u^{\mu}(x_{\text{in}}) \quad (21)$$

As for the hydrodynamic pressure and equation of state (EOS), $p_{\text{hyd}} = p_{\text{hyd}}(\epsilon_{\text{hyd}})$, they can be chosen, for example, in agreement with the lattice QCD EOS for the RHIC and LHC energies, and the same for the viscous coefficients. Then one can use Eq. (19) with the initial conditions (21) as phenomenological macroscopic equations for a description of transient pre-hydrodynamic behavior in ultrarelativistic heavy-ion collisions.

III. A SIMPLE MODEL TO ASSESS THE INITIAL CONDITIONS FOR HYDRODYNAMICS

Let us begin with the example, which we choose merely because of its simplicity, where the initial and final conditions are specified at constant-time hypersurfaces $t = t_0$ and $t = t_{\text{th}}$, respectively. Then, in obvious notations,

$$p^{\mu} \frac{\partial}{\partial x^{\mu}} = p_0 \frac{\partial}{\partial t} + \mathbf{p} \frac{\partial}{\partial \mathbf{r}}, \quad (22)$$

and we get from (13) and (22) that

$$G_p(x - x') = p_0^{-1} \Theta(t - t') \Theta(t' - t_0) \delta^{(3)}(\mathbf{r}(t, t') - \mathbf{r}'), \quad (23)$$

where $\Theta(t' - t_0)$ indicates that the evolution time starts at t_0 , and

$$\mathbf{r}(t, t') = \mathbf{r} - (\mathbf{p}/p_0)(t - t') \quad (24)$$

satisfies the equation

$$p^{\mu} \partial_{\mu} \mathbf{r}(t, t') = 0. \quad (25)$$

Substituting these results into (12) and integrating over \mathbf{r}' we get an expression that explicitly coincides with (8):

$$\mathcal{P}(t, \mathbf{r}, p) = \exp \left\{ - \int_{t_0}^t \frac{1}{t_{\text{rel}}(s, \mathbf{r}(t, s), p)} ds \right\}, \quad (26)$$

where

$$t_{\text{rel}}(s, \mathbf{r}(t, s), p) = \frac{p_0 \tau_{\text{rel}}^*(s, \mathbf{r}(t, s), p)}{p^\mu u_\mu(s, \mathbf{r}(t, s))}. \quad (27)$$

Referring to Eqs. (11), (22) and (25), we see that

$$f_{\text{free}}(x, p) = f(t_0, \mathbf{r}(t_0), p). \quad (28)$$

Equation (28) allows to build the evolution of $T_{\text{free}}^{\mu\nu}$ and after specification of τ_{rel}^* to fix the equation of relaxation (19). In the next example, which is very important in practice, we analyze this procedure in detail.

Now let us consider the hypersurfaces $\tau = \tau_0$ and $\tau = \tau_{\text{th}}$ as the initial and final ("thermalization") hypersurfaces, respectively; here $\tau = \sqrt{t^2 - z^2}$ is the longitudinal proper time. In order to describe the boost-invariant dynamics, it is convenient to switch to the coordinates $(\tau, \mathbf{r}_T, \eta)$, where \mathbf{r}_T is the transverse radius vector and η is the space-time rapidity, $\eta = \tanh^{-1} z/t$; then $t = \tau \cosh \eta$ and $z = \tau \sinh \eta$. The particle four-momentum can be expressed through the momentum rapidity $y = \tanh^{-1} p_L/p_0$ where p_L is the longitudinal momentum, the transverse momentum \mathbf{p}_T , and the transverse mass $m_T = \sqrt{m^2 + p_T^2}$, then $p^\mu = (m_T \cosh y, \mathbf{p}_T, m_T \sinh y)$. Accounting for the boost-invariant dynamics, we choose

$$\tau_{\text{rel}}^*(x, p) = \tau_{\text{rel}}^*(\tau, \theta, \mathbf{r}_T, \mathbf{p}_T), \quad (29)$$

where $\theta = y - \eta$. Transforming $p^\mu \partial_\mu$ to these coordinates, we find

$$p^\mu \frac{\partial}{\partial x^\mu} = m_T \cosh \theta \frac{\partial}{\partial \tau} + \frac{m_T}{\tau} \sinh \theta \frac{\partial}{\partial \eta} + \mathbf{p}_T \frac{\partial}{\partial \mathbf{r}_T}. \quad (30)$$

The auxiliary function G_p then reads

$$G_p(x - x') = \frac{\Theta(\tau - \tau') \Theta(\tau' - \tau_0) \delta(\eta(\tau, \tau') - \eta') \delta^{(2)}(\mathbf{r}_T(\tau, \tau') - \mathbf{r}')}{m_T \cosh(y - \eta')}, \quad (31)$$

where $\Theta(\tau' - \tau_0)$ indicates that the time evolution starts at nonzero proper time τ_0 . Here $\eta(\tau, \tau')$ and $\mathbf{r}_T(\tau, \tau')$ satisfy the equations

$$p^\mu \partial_\mu \eta(\tau, \tau') = 0, \quad (32)$$

$$p^\mu \partial_\mu \mathbf{r}_T(\tau, \tau') = 0, \quad (33)$$

and can be written as

$$\sinh \theta(\tau, \tau') = \frac{\tau}{\tau'} \sinh \theta, \quad (34)$$

$$\mathbf{r}_T(\tau, \tau') = \mathbf{r}_T - \frac{\mathbf{p}_T}{m_T}(\tau \cosh \theta - \sqrt{\tau'^2 + \tau^2 \sinh^2 \theta}), \quad (35)$$

where $\theta(\tau, \tau') = y - \eta(\tau, \tau')$. Substituting (31) into (12), accounting for (29), and integrating over η' and \mathbf{r}'_T , we get

$$\mathcal{P}(\tau, \theta, \mathbf{r}_T, \mathbf{p}_T) = \exp \left(- \int_{\tau_0}^{\tau} \frac{1}{\tau_{\text{rel}}(\tau', \theta(\tau, \tau'), \mathbf{r}_T(\tau, \tau'), \mathbf{p}_T)} d\tau' \right), \quad (36)$$

where

$$\tau_{\text{rel}}(\tau', \theta(\tau, \tau'), \mathbf{r}_T(\tau, \tau'), \mathbf{p}_T) = \frac{m_T \cosh \theta(\tau, \tau')}{p^\mu u_\mu(\tau', \theta(\tau, \tau'), \mathbf{r}_T(\tau, \tau'), \mathbf{p}_T)} \tau_{\text{rel}}^*(\tau', \theta(\tau, \tau'), \mathbf{r}_T(\tau, \tau'), \mathbf{p}_T). \quad (37)$$

Then $f_{\text{free}}(x, p)$ is given by

$$f_{\text{free}}(x, p) = f(\tau_0, \theta(\tau, \tau_0), \mathbf{r}_T(\tau, \tau_0), \mathbf{p}_T). \quad (38)$$

In order to pass now to a description of the relaxation dynamics at the prethermal stage, let us take into account that at this stage the longitudinal flow is much stronger than the transverse one and so, according to Eq. (37), $\tau_{\text{rel}} \approx \tau_{\text{rel}}^*$. Therefore, to get a convenient parametrization of the prethermal evolution of the energy-momentum tensor of the system, one might assume that the relaxation time τ_{rel}^* in the rest frames of the fluid elements depends mainly only on the proper time τ : $\tau_{\text{rel}}^* = \tau_{\text{rel}}^*(\tau)$. Then \mathcal{P} also depends on τ only. It corresponds to the Bjorken picture [24], where the thermalization processes are supposed to be synchronous in proper time of the fluid elements, so that the complete thermalization and the beginning of the hydrodynamic expansion happen at some common proper time τ_{th} . Within such an approximation we get

$$T^{\mu\nu}(x) = T_{\text{free}}^{\mu\nu}(x) \mathcal{P}(\tau) + T_{\text{hyd}}^{\mu\nu}(x) (1 - \mathcal{P}(\tau)), \quad (39)$$

where

$$T_{\text{free}}^{\mu\nu}(x) = \int d^2 p_T d\theta p^\mu p^\nu f(\tau_0, \theta(\tau, \tau_0), \mathbf{r}_T(\tau, \tau_0), \mathbf{p}_T), \quad (40)$$

and $T_{\text{hyd}}^{\mu\nu}(x)$ is associated with ideal or viscous hydrodynamics; for example, for the former it can be parametrized as follows:

$$T_{\text{hyd}}^{\mu\nu}(x) = (\epsilon_{\text{hyd}}(x) + p_{\text{hyd}}(x)) u_{\text{hyd}}^\mu(x) u_{\text{hyd}}^\nu(x) - p_{\text{hyd}}(x) g^{\mu\nu}, \quad (41)$$

where $\epsilon_{\text{hyd}}(x)$ is the energy density in the comoving system and $p_{\text{hyd}}(x)$ is the pressure. The corresponding evolutional equations are⁸

$$\partial_\mu[(1 - \mathcal{P}(\tau))T_{\text{hyd}}^{\mu\nu}(x)] = -T_{\text{free}}^{\mu\nu}(x)\partial_\mu\mathcal{P}(\tau). \quad (42)$$

In order to link Eqs. (39) and (42) with the hydrodynamics at $\tau = \tau_{\text{th}}$, one needs

$$\mathcal{P}(\tau_0) = 1, \quad \mathcal{P}(\tau_{\text{th}}) = 0, \quad \partial_\mu\mathcal{P}(\tau_{\text{th}}) = 0. \quad (43)$$

The initial conditions for $T_{\text{free}}^{\mu\nu}$ coincide with initial conditions for $T^{\mu\nu}$, namely $T_{\text{free}}^{\mu\nu}(\tau_0, \mathbf{r}_T, \eta) = T^{\mu\nu}(\tau_0, \mathbf{r}_T, \eta)$. The initial conditions for $T_{\text{hyd}}^{\mu\nu}$ (see Eq. (20)) are defined by Eq. (21). The hydrodynamic pressure and equation of state, $p_{\text{hyd}} = p_{\text{hyd}}(\epsilon_{\text{hyd}})$, can be chosen, for example, in agreement with the lattice QCD EOS.

In order to apply the equation of relaxation dynamics (42) to calculate the space-time evolution of the energy-momentum tensor toward the hydrodynamic one, the function $\mathcal{P}(\tau)$ has to be specified. Without a knowledge of the specific thermalization dynamics, we propose to make the approximation for this as simple as possible with a minimal set of parameters. With that end in view, we do not discuss here the most general case, but propose the following simple ansatz for $\mathcal{P}(\tau)$:

$$\mathcal{P}(\tau) = \exp \left\{ - \int_{\tau_0}^{\tau} \frac{1}{\tau_{\text{rel}}(s)} ds \right\}, \quad (44)$$

where

$$\tau_{\text{rel}}(s) = \tau_{\text{rel}}(\tau_0) \frac{\tau_{\text{th}} - s}{\tau_{\text{th}} - \tau_0}. \quad (45)$$

Performing the integral in (44), we find that

$$\mathcal{P}(\tau) = \left(\frac{\tau_{\text{th}} - \tau}{\tau_{\text{th}} - \tau_0} \right)^{\frac{\tau_{\text{th}} - \tau_0}{\tau_{\text{rel}}(\tau_0)}}. \quad (46)$$

⁸ Note that Eq. (42) can be considered in computational dynamics as the hydrodynamic equation for the energy-momentum tensor $\tilde{T}_{\text{hyd}}^{\mu\nu} = (1 - \mathcal{P}(\tau))T_{\text{hyd}}^{\mu\nu}$ of an ideal fluid with an explicit "source" term on the right-hand side and with rescaled energy density $\tilde{\epsilon}_{\text{hyd}} = (1 - \mathcal{P}(\tau))\epsilon_{\text{hyd}}$, and pressure $\tilde{p}_{\text{hyd}} = (1 - \mathcal{P}(\tau))p_{\text{hyd}}$. Also, since in our approximation the relaxation time and, hence, the probability \mathcal{P} of freely propagation are supposed to be known functions (up to the two fitting parameters), as well as the tensor of free propagating system $T_{\text{free}}^{\mu\nu}(x)$, then Eqs. (42) are of the same type as the equations $\partial_\mu T_{\text{hyd}}^{\mu\nu}(x) = 0$. So, if the latter are equations of the hyperbolic type, as in the case of the ideal fluid or second-order viscous hydrodynamics (in the form of Israel and Stewart), then causality is preserved in our approach.

The conditions (43) require $\frac{\tau_{\text{th}} - \tau_0}{\tau_{\text{rel}}(\tau_0)} > 1$ (then also $\mathcal{P}/\tau_{\text{rel}}^* \rightarrow 0$ at $\tau \rightarrow \tau_{\text{th}}$) which is a constraint on the model parameters: the lifetime of the prehydrodynamical period $(\tau_{\text{th}} - \tau_0)$ and the initial value of the relaxation-time parameter $\tau_{\text{rel}}(\tau_0)$. The energy-momentum relaxation equation (42) together with the proposed parametrization (46) for the probability function $\mathcal{P}(\tau)$ can be used to assess $T^{\mu\nu}(\tau_{\text{th}}, \mathbf{r}_T, \eta) = T_{\text{hyd}}^{\mu\nu}(\tau_{\text{th}}, \mathbf{r}_T, \eta)$ and so find the initial conditions (in particular, the energy density and transverse flow) for hydrodynamic simulations of ultrarelativistic heavy-ion collisions.

The heuristic method, which allows us to derive these macroscopic equations, is based on the Boltzmann equation in the relaxation-time approximation, but, to our mind, its applicability area is not restricted to a rarefied system just as the form of the hydrodynamic equations based on approximate solutions of the Boltzmann equations near the local equilibrium distribution can be applied for a description of dense fluids if they are near a local thermal equilibrium. It is worth noting that Eqs. (42), (44), and (45) do not contain details of the system's properties, except for the initial energy-momentum tensor and its free evolution. The method makes it possible to match the initial nonequilibrium state of the system and the locally equilibrated one for various kinds of system, such as partons, strings, and a glasma field. It also allows one to match a far-from-equilibrium initial state not only to the locally equilibrated one but also to the state associated with the initial conditions for viscous hydrodynamic evolution of the QGP. For the latter, Eqs. (42) and (46) preserve their form, but the energy-momentum tensor $T_{\text{hyd}}^{\mu\nu}(x)$ differs from the simple form of perfect hydrodynamics (41) and corresponds to the tensor of viscous hydrodynamics. It should be noted that in this case Eq. (45) loses its interpretation as the relaxation time to the local equilibrium state (it tends to zero at $\tau \rightarrow \tau_{\text{th}}$) and so the probability (46) plays now just the role of the interpolating function. Nevertheless, it allows one to match smoothly the very initial state of the matter in $A + A$ collisions with the initial conditions for viscous hydrodynamics when (partial) thermalization is already established; in other words, to find the initial hydrodynamic parameters - energy density, hydrodynamic velocity field, and so on - for the dissipative evolution of the QGP in agreement with the model relaxation dynamics and the conservation laws utilized at the prethermal stage.

IV. SUMMARY

In this paper we presented a simple model of the early stage in ultrarelativistic heavy-ion collisions. The model describes smooth conversion of the preequilibrium dynamics into hydrodynamics, either perfect or viscous. Our phenomenological approach is motivated by the Boltzmann equation in the relaxation-time approximation, accounts for the energy and momentum conservation laws, and contains two parameters: the lifetime of the prehydrodynamic stage and the initial value of the relaxation-time parameter. The preequilibrium evolution is modeled by the continuous evolution of the energymomentum tensor from the initial far-from-equilibrium state to the perfect or viscous fluid form. The model allows the flows and energy densities to be assessed at a supposed time of initialization of hydrodynamics, which then can be used as the initial condition for hydrodynamic simulations of the further evolution of matter (QGP) in ultrarelativistic heavy-ion collisions.

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